

GENERALIZATIONS OF ANALOGS OF THEOREMS OF MAIZEL AND PLISS AND THEIR APPLICATION IN SHADOWING THEORY

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ABSTRACT. We generalize two classical results of Maizel and Pliss that describe relations between hyperbolicity properties of linear system of difference equations and its ability to have a bounded solution for every bounded inhomogeneity. We also apply one of this generalizations in shadowing theory of diffeomorphisms to prove that some sort of limit shadowing is equivalent to structural stability.

1. INTRODUCTION

In [10] Perron defined property (B) for systems of differential equations. The property is that an inhomogeneous system of differential equations has bounded solution for every bounded inhomogeneity. In [7] A. Maizel proved a theorem that links property (B) on the half-line with the hyperbolicity property. In [16] Pliss characterized an analog of the property (B) on the full line in terms of hyperbolicity on two half-lines. The proof of the Pliss' theorem is based on the Maizel's theorem.

The property (B) is often called admissibility. There exist many papers devoted to study of this property and its analogs. For references see [1, 6, 5].

We prove generalizations of both theorems for difference equations for the case of spaces of sequences with prescribed decay rate for the case of finite dimensions and bounded coefficients. All similar theorems that were proved so far deal only with spaces having certain homogeneity properties.

The discrete analog of the Pliss' theorem is widely used in shadowing theory (see [14, 18, 15]). As an application of its generalized version we introduce the notion of Lipschitz two-sided limit shadowing property and prove that this property is equivalent to the structural stability.

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2. DEFINITIONS

Let I be either $\mathbb{Z}^+ = \{k \in \mathbb{Z} \mid k \geq 0\}$ or $\mathbb{Z}^- = \{k \in \mathbb{Z} \mid k \leq 0\}$ or \mathbb{Z} . Let $\mathcal{A} = \{A_k\}_{k \in I}$ be a sequence of linear isomorphisms $\mathbb{R}^d \rightarrow \mathbb{R}^d$ indexed by integers from I . Consider homogeneous and inhomogeneous equations associated with this sequence.

$$x_{k+1} = A_k x_k, \quad k \in I \quad (2.1)$$

$$x_{k+1} = A_k x_k + f_{k+1}, \quad k \in I. \quad (2.2)$$

Remark 2.1. For $I = \mathbb{Z}^+$ we take f_k to be defined for $k \geq 0$ and $f_0 = 0$.

Define an analog of fundamental matrix for equations (2.1):

$$\Phi_{m,l} = \begin{cases} A_{m-1} \circ \dots \circ A_l, & m > l, \\ Id, & m = l, \\ A_m^{-1} \circ \dots \circ A_{l-1}^{-1}, & m < l. \end{cases}$$

We fix $\omega \geq 0$. We use linear subspaces of the space of sequences of vectors from \mathbb{R}^d , indexed by integers from I . Denote the Banach space of sequences with bounded norm $\|x\|_\omega = \sup_{k \in I} |x_k| (|k| + 1)^\omega$ by $\mathcal{N}_\omega(I)$. Such spaces has been already studied in the similar context (see [2]). It is important to note that these spaces are neither homogeneous in the sense of Baskakov (see [1]) nor translation-Invariant in the sense of Sasu (see [17]).

Definition 1. We say that a sequence \mathcal{A} has Perron property $B_\omega(I)$ if for any sequence $f \in \mathcal{N}_\omega(I)$ there exists a solution of the inhomogeneous system of difference equations with inhomogeneity f that belongs to $\mathcal{N}_\omega(I)$.

We use the following definition from [11]:

Definition 2. We say that a sequence \mathcal{A} is hyperbolic on I if there exist constants $K > 0$, $\lambda \in (0, 1)$ and projections P_k, Q_k , $k \in I$ such that if $S_k = P_k \mathbb{R}^d$ and $U_k = Q_k \mathbb{R}^d$ then the following holds:

$$\mathbb{R}^d = S_k \oplus U_k; \quad (2.3)$$

$$A_k S_k = S_{k+1}, \quad A_k U_k = U_{k+1}; \quad (2.4)$$

$$|\Phi_{k,l} v| \leq K \lambda^{k-l} |v|, \quad v \in S_l, \quad k \geq l; \quad (2.5)$$

$$|\Phi_{k,l} v| \leq K \lambda^{l-k} |v|, \quad v \in U_l, \quad k \leq l; \quad (2.6)$$

$$\|P_k\|, \|Q_k\| \leq K. \quad (2.7)$$

Everywhere here we mean that all indices are from I .

Remark 2.2. We call the spaces S_k and U_k stable and unstable spaces of the sequence \mathcal{A} .

Remark 2.3. If norms of all $\|A_k\|$ and $\|A_k\|^{-1}$ are bounded then conditions (2.5) and (2.6) imply condition (2.7) (with different constant K in general), which is equivalent to the boundedness from zero of the angle between the spaces S_k and U_k (see [4] p. 224, 234, 237 for example).

Remark 2.4. Let $I = \mathbb{Z}^+$. Then conditions (2.5) and (2.6) for some $\lambda \in (0, 1)$ and $K > 0$ follow from the existence of $\lambda_1 \in (0, 1)$ and $K_1 > 0$ such that the following

estimates hold

$$|\Phi_{k,l}v| \leq K_1 \lambda_1^{k-l} (k+1)^{-\omega} (l+1)^\omega |v|, \quad v \in S_l, \quad k \geq l; \quad (2.8)$$

$$|\Phi_{k,l}v| \leq K_1 \lambda_1^{l-k} (k+1)^{-\omega} (l+1)^\omega |v|, \quad v \in U_l, \quad k \leq l. \quad (2.9)$$

Also everywhere here we mean that all indices are from I .

Proof. Let conditions (2.8) and (2.9) be satisfied. We show that conditions (2.5) and (2.6) are also satisfied:

$$\begin{aligned} K_1 \lambda_1^{l-k} (k+1)^{-\omega} (l+1)^\omega &\leq K_1 \lambda_1^{\frac{l}{2}} (l+1)^\omega \leq \\ &\leq \left(\max_{l \in \mathbb{Z}^+} \left(K_1 \lambda_1^{\frac{l}{2}} (l+1)^\omega \right) \right) \left(\lambda_1^{\frac{1}{2}} \right)^l = K_2 \lambda_2^l \leq K_2 \lambda_2^{l-k}, \quad 2k \leq l \end{aligned}$$

and

$$\begin{aligned} K_1 \lambda_1^{l-k} (k+1)^{-\omega} (l+1)^\omega &\leq K_1 \lambda_1^{l-k} (k+1)^{-\omega} (2k+1)^\omega = \\ &= 2^\omega K_1 \lambda_1^{l-k} \left(\frac{k+\frac{1}{2}}{k+1} \right)^\omega \leq K_3 \lambda_1^{l-k}, \quad 2k \geq l \geq k. \end{aligned}$$

This proves inequality (2.6) for $\lambda = \max(\lambda_1, \lambda_2)$ and $K = \max(K_1, K_2, K_3)$. Inequality (2.5) is obvious since

$$(k+1)^{-\omega} (l+1)^\omega \leq 1, \quad k \geq l.$$

□

3. MAIN RESULTS

We prove the following theorem in Section 4.1:

Theorem 1 (a generalization of the discrete analog of Maizel Theorem). *Let $I = \mathbb{Z}^+$ and the norms of all matrices A_k and A_k^{-1} be bounded by $M > 0$. A sequence \mathcal{A} has property $B_\omega(I)$ iff it is hyperbolic on \mathbb{Z}^+ .*

We prove the following theorem in Section 4.2:

Theorem 2 (a generalization of the discrete analog of Pliss Theorem). *Let $I = \mathbb{Z}^+$ and the norms of all matrices A_k and A_k^{-1} be bounded by $M > 0$. A sequence \mathcal{A} has property $B_\omega(I)$ iff it is hyperbolic on both \mathbb{Z}^+ and \mathbb{Z}^- and the spaces $B^+(\mathcal{A})$ and $B^-(\mathcal{A})$ are transverse. Here*

$$\begin{aligned} B^+(\mathcal{A}) &= \{v \in \mathbb{R}^d \mid |\Phi_{k,0}v| \rightarrow 0, \quad k \rightarrow +\infty\}, \\ B^-(\mathcal{A}) &= \{v \in \mathbb{R}^d \mid |\Phi_{k,0}v| \rightarrow 0, \quad k \rightarrow -\infty\}. \end{aligned}$$

4. GENERALIZATIONS OF DISCRETE ANALOGS OF THEOREMS OF MAIZEL AND PLISS

We prove generalizations of theorems of Maizel and Pliss for the case of difference equations.

4.1. Maizel Theorem. Let $I = \mathbb{Z}^+$. For brevity we write \mathcal{N}_ω instead of $\mathcal{N}_\omega(I)$. Assume that the sequence \mathcal{A} has property $B_\omega(I)$.

Denote

$$V_1 = \{x_0 \mid x \in \mathcal{N}_\omega, x \text{ is a solution of homogeneous equation (2.1)}\}.$$

Since our equations are linear and \mathcal{N}_ω is a linear space, V_1 is also a linear space. Denote the orthogonal complement of V_1 by V_2 and orthogonal projection onto V_1 by P .

It is easy to see that the following holds:

Statement 4.1. *For any sequence $f \in \mathcal{N}_\omega$ there exists only one solution $T(f) \in \mathcal{N}_\omega$ of inhomogeneous equation (2.2) with inhomogeneity f such that $(T(f))_0 \in V_2$.*

Statement 4.2. *For any sequence \mathcal{A} the operator $T : \mathcal{N}_\omega \rightarrow \mathcal{N}_\omega$ from the previous statement is continuous. In particular there exists a positive r such that*

$$\|Tf\|_\omega \leq r \|f\|_\omega.$$

Proof. Fully analogous to the proof of Statement 4 from [3]. \square

From now on we use the operator T and number r from the previous statement. Also we suppose that $r \geq 1$ and that the number M from the statement of Theorem 1 satisfies inequality $rM \geq 1$.

4.1.1. Technical lemmas. Denote

$$X_k = \begin{cases} \Phi_{k,0}, & k > 0, \\ Id, & k = 0, \\ \Phi_{0,-k}, & k < 0. \end{cases}$$

It is easy to see that the following holds.

Statement 4.3. *The following formula provides a solution of inhomogeneous difference equation (2.2):*

$$(4.1) \quad y_k = \sum_{u=0}^k X_k P X_{-u} f_u - \sum_{u=k+1}^{\infty} X_k (I - P) X_{-u} f_u,$$

when the series in the second summand converges. Here we take f_0 equal to 0.

Remark 4.4. We do not care about exact values of the constants and always assume that the value of M is big enough and neglect constants like one that bounds from above the sequence $(1 - 1/k)^\beta$ for real β .

Remark 4.5. The following formula can be interpreted as an analog of the Green's function for difference equations

$$G_{k,u} = \begin{cases} X_k P X_{-u}, & 0 \leq u \leq k, \\ -X_k (I - P) X_{-u}, & 0 \leq k < u. \end{cases}$$

So formula (4.1) can be rewritten in a more compact way:

$$(4.2) \quad y_k = \sum_{u=0}^{\infty} G_{k,u} f_u.$$

Lemma 4.6. *Let k_0, k_1, k be nonnegative integers and $\xi \in \mathbb{R}^d$ be a nonzero vector. Then the following inequalities hold*

$$|X_k P \xi| \sum_{u=k_0}^k (u+1)^{-\omega} |X_u \xi|^{-1} \leq r(k+1)^{-\omega}, \quad 0 \leq k_0 \leq k, \quad (4.3)$$

$$|X_k (I - P) \xi| \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} \leq 2rM(k+1)^{-\omega}, \quad 0 \leq k \leq k_1. \quad (4.4)$$

Proof. Fix nonnegative integers l_0, l_1 such that $l_0 \leq l_1$. Consider a sequence f with $f_i = 0$, $i > l_1$. Then formula (4.2) looks like:

$$y_l = \sum_{u=0}^{l_1} G_{l,u} f_u.$$

For $l \geq l_1$ all the indices u in the sum are less or equal than l_1 and the first string from the definition of $G_{l,u}$ is used. The previous equality turn into the following:

$$y_l = X_l P \sum_{u=0}^{l_1} X_{-u} f_u.$$

Thus the vector y_l for $l \geq l_1$ is an image of the vector from V_1 that is independent of l . This means that all the sequence y except a finite number of entries is a solution of homogeneous equation (2.1) with initial conditions from V_1 . Thus y belongs to \mathcal{N}_ω . Using that $f_0 = 0$ we obtain

$$y_0 = -(I - P) \sum_{u=0}^{l_1} X_{-u} f_u \in V_2.$$

So $y = Tf$ and therefore $\|y\|_\omega \leq r \|f\|_\omega$. Let $x_i = X_i \xi$. We define the sequence f :

$$f_i = \begin{cases} 0, & i < l_0, \\ (i+1)^{-\omega} \frac{x_i}{|x_i|}, & l_0 \leq i \leq l_1, \\ 0, & i > l_1. \end{cases}$$

Then $\|f\|_\omega = 1$. Substituting the formula for a solution in the inequality from Statement 4.2 we obtain

$$(4.5) \quad \left| \sum_{u=l_0}^{l_1} (u+1)^{-\omega} G_{l,u} \frac{x_u}{|x_u|} \right| = |y_l| \leq r(l+1)^{-\omega}.$$

For $l_1 = l = k$, $l_0 = k_0$ from (4.5) we obtain that if $k \geq k_0$ then

$$\begin{aligned} r(k+1)^{-\omega} &\geq \left| \sum_{u=k_0}^k (u+1)^{-\omega} G_{k,u} \frac{x_u}{|x_u|} \right| = \left| \sum_{u=k_0}^k (u+1)^{-\omega} X_k P X_{-u} \frac{X_u \xi}{|X_u \xi|} \right| = \\ &= \left| X_k P \xi \sum_{u=k_0}^k (u+1)^{-\omega} |X_u \xi|^{-1} \right| = |X_k P \xi| \sum_{u=k_0}^k (u+1)^{-\omega} |X_u \xi|^{-1}. \end{aligned}$$

To prove the second inequality from the statement of the lemma we do the similar. The important thing to notion here is that in the second string of the definition

of $G_{k,s}$ the inequality is strict. Then for $l = k - 1$, $l_0 = k$, $l_1 = k_1$ from (4.5) we obtain that for $0 < k \leq k_1$ we have

$$\begin{aligned} rk^{-\omega} &\geq \left| \sum_{u=k}^{k_1} (u+1)^{-\omega} G_{k-1,u} \frac{x_u}{|x_u|} \right| = \left| \sum_{u=k}^{k_1} -(u+1)^{-\omega} X_{k-1} (I-P) X_{-u} \frac{X_u \xi}{|X_u \xi|} \right| = \\ &= \left| X_{k-1} (I-P) \xi \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} \right| = |X_{k-1} (I-P) \xi| \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} = \\ &= |A_{k-1}^{-1} X_k (I-P) \xi| \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} \geq \|A_{k-1}\|^{-1} |X_k (I-P) \xi| \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1}. \end{aligned}$$

Now we prove the second inequality of the statement of the lemma for the case when $0 = k < k_1$ using the previous inequality for $k = 1$:

$$\begin{aligned} |X_0 (I-P) \xi| \sum_{u=0}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} &= |X_0 (I-P) \xi| \sum_{u=1}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} + |(I-P) \xi| \leq \\ &\leq \|A_0^{-1}\| |X_1 (I-P) \xi| \sum_{u=1}^{k_1} (u+1)^{-\omega} |X_u \xi|^{-1} + 1 \leq r \|A_0^{-1}\| + 1. \end{aligned}$$

For $k = k_1 = 0$ the inequality is obvious. \square

Lemma 4.7. *Let k_0, k_1, k, s be nonnegative integers and ξ be a vector.*

Denote

$$\mu = 1 - (2rM)^{-1}.$$

The following inequalities are satisfied:

if $P\xi \neq 0$ then

$$\sum_{u=k_0}^s (u+1)^{-\omega} |X_u P\xi|^{-1} \leq \mu^{k-s} \sum_{u=k_0}^k (u+1)^{-\omega} |X_u P\xi|^{-1} \quad (4.6)$$

for $k \geq s \geq k_0$;

if $(I-P)\xi \neq 0$ then

$$\sum_{u=s}^{k_1} (u+1)^{-\omega} |X_u (I-P)\xi|^{-1} \leq \mu^{s-k} \sum_{u=k}^{k_1} (u+1)^{-\omega} |X_u (I-P)\xi|^{-1} \quad (4.7)$$

for $k_1 \geq s \geq k$;

Proof. Denote

$$\begin{aligned} \phi_i &= \sum_{u=k_0}^i (u+1)^{-\omega} |X_u P\xi|^{-1}, \quad i \geq k_0. \\ \psi_i &= \sum_{u=i}^{k_1} (u+1)^{-\omega} |X_u (I-P)\xi|^{-1}, \quad i \leq k_1, \end{aligned}$$

We prove inequality (4.6). Since $P\xi \neq 0$, it is easy to see that $\phi_k > 0$.

Also it is obvious that $\phi_k - \phi_{k-1} = (k+1)^{-\omega} |X_k P\xi|^{-1}$. Thus replacing ξ by $P\xi$ in (4.3) we get

$$\frac{\phi_k}{\phi_k - \phi_{k-1}} \leq r \leq 2rM.$$

Then

$$(2rM)^{-1} \leq \frac{\phi_k - \phi_{k-1}}{\phi_k} = 1 - \frac{\phi_{k-1}}{\phi_k}.$$

Therefore $\phi_{k-1} \leq (1 - (2rM)^{-1})\phi_k$. If we consequently use this inequality enough times, we obtain

$$\begin{aligned} \phi_s &\leq (1 - (2rM)^{-1}) \dots (1 - (2rM)^{-1}) \phi_k = \\ &= (1 - (2rM)^{-1})^{k-s} \phi_k, \quad k \geq s. \end{aligned}$$

To prove the second inequality from the statement of the lemma recall that $\psi_k > 0$. Analogous to the proof of the first inequality from the statement of the lemma, $\psi_k - \psi_{k+1} = (k+1)^{-\omega} |X_k(I-P)\xi|^{-1}$ and replacing ξ by $(I-P)\xi$ in (4.4) we have

$$\psi_{k+1} \leq \left(1 - (2rM)^{-1} k^\omega (k+1)^{-\omega}\right) \psi_k.$$

Again, if we consequently use this inequality enough times and use Remark 4.4 we get inequality (4.7). \square

4.1.2. Proof of the discrete analog of the Maizel theorem.

Theorem 3. *The following inequalities holds*

$$\|X_k P X_{-s}\| \leq r^2 (k+1)^{-\omega} (s+1)^\omega \mu^{k-s}$$

for $0 \leq s \leq k$;

$$\|X_k (I-P) X_{-s}\| \leq 2r^2 M^2 (k+1)^{-\omega} (s+1)^\omega \mu^{s-k}$$

for $0 \leq k < s$.

Proof. Fix a natural $s \geq 1$ and a unit vector ξ . Define a sequence y :

$$y_k = \begin{cases} -X_k (I-P) X_{-s} \xi, & 0 \leq k < s, \\ X_k P X_{-s} \xi, & k \geq s. \end{cases}$$

The sequence y coincides (except a finite number of entries) with a solution of homogenous equation (2.1) with initial conditions from V_1 and therefore y belongs to \mathcal{N}_ω . Now we define a sequence f in such a way as the sequence y is a solution of inhomogeneous equation (2.2) with inhomogeneity f :

$$f_k = \begin{cases} 0, & k \neq s, \\ \xi, & k = s. \end{cases}$$

It is easy to see that in this case y becomes a solution. This means that $y = Tf$. Thus $\|y\|_\omega \leq r \|f\|_\omega = r(s+1)^\omega$. We prove the first inequality for the operator norms from the statement of the theorem. Using the definition of the sequence y we can write

$$|X_k P X_{-s} \xi| = |y_k| \leq r(k+1)^{-\omega} (s+1)^\omega, \quad s \leq k.$$

Since ξ can be any unit vector, this gives us an estimate for the operator norms of $X_k P X_{-s}$. Now we can replace x by the solution of the homogeneous equation $x_k = X_k \xi$ and substitute in the previous inequality instead of ξ :

$$(4.8) \quad |X_k P \xi| = |X_k P X_{-s} x_s| \leq r(k+1)^{-\omega} (s+1)^\omega |x_s|, \quad s \leq k.$$

Let $P\xi \neq 0$. Consequently using inequalities (4.3) and (4.6) for $k_0 = s$ and (4.8) for $k = s$ we get

$$\begin{aligned} |X_k P X_{-s} x_s| &= |X_k P \xi| \leq r(k+1)^{-\omega} \left(\sum_{u=s}^k (u+1)^{-\omega} |X_u P \xi|^{-1} \right)^{-1} \leq \\ &\leq r(k+1)^{-\omega} \left(\mu^{-(k-s)} (s+1)^{-\omega} |X_s P \xi|^{-1} \right)^{-1} = \\ &= r(k+1)^{-\omega} \mu^{k-s} (s+1)^{\omega} |X_s P \xi| \leq \\ &\leq r^2 (k+1)^{-\omega} (s+1)^{\omega} \mu^{k-s} |x_s|. \end{aligned}$$

If $P\xi = 0$ then the resulting inequality is obvious. Since $x_s = X_s \xi$ and X_s is an isomorphism, we have an estimate for the operator norm:

$$\|X_k P X_{-s}\| \leq r^2 (k+1)^{-\omega} (s+1)^{\omega} \mu^{k-s}, \quad 1 \leq s \leq k.$$

In this reasoning we have used only the fact that inequality (4.8) is satisfied for $s = k$. This is also true for $s = k = 0$ since $\|P\| \leq 1$. Therefore, we proved the estimate for $0 \leq s \leq k$.

The proof of the second estimate from the statement for $s > k$ is similar to the proof of the first estimate. The only small differences are due to the fact that now we cannot use an analog of inequality (4.8) for $k = s$ because in the definition of the sequence y the numbers s and k cannot be equal. The following inequality can be proved in a very same manner as one in the proof of the first estimate:

$$|X_k (I - P) \xi| = |X_k (I - P) X_{-s} x_s| \leq r(k+1)^{-\omega} (s+1)^{\omega} |x_s|, \quad s > k.$$

For $k = s - 1$ we multiply the vector inside the norm brackets by A_{s-1} :

$$\begin{aligned} |X_s (I - P) \xi| &= |A_{s-1} X_{s-1} (I - P) X_{-s} x_s| \leq \\ (4.9) \quad &\leq \|A_{s-1}\| |X_{s-1} (I - P) X_{-s} x_s| \leq rM(s+1)^{\omega} s^{-\omega} |x_s|. \end{aligned}$$

After that the proof is fully analogous to the proof of the first estimate. \square

Lemma 4.8. *For any $\lambda \in (0, 1)$ there exists a constant $C > 0$ depending only on λ and ω such that*

$$(4.10) \quad (k+1)^{\omega} \left(\sum_{u=0}^k \lambda^{k-u} (u+1)^{-\omega} + \sum_{u=k+1}^{\infty} \lambda^{u-k} (u+1)^{-\omega} \right) < C$$

for any $k \geq 0$.

Proof. We estimate first summand from (4.10). To do this it is enough to estimate the corresponding integral:

$$\begin{aligned} (k+1)^{\omega} \int_0^k \lambda^{k-u} (u+1)^{-\omega} du &= \\ &= (k+1)^{\omega} \left(\int_0^{\frac{k}{2}} \lambda^{k-u} (u+1)^{-\omega} du + \int_{\frac{k}{2}}^k \lambda^{k-u} (u+1)^{-\omega} du \right). \end{aligned}$$

Now we estimate separately the two integrals from the previous formula. The first one can be estimated in the following way:

$$(k+1)^{\omega} \int_0^{\frac{k}{2}} \lambda^{k-u} (u+1)^{-\omega} du \leq (k+1)^{\omega} \int_0^{\frac{k}{2}} \lambda^{k-u} du \leq$$

$$\leq (k+1)^\omega \int_0^{\frac{k}{2}} \lambda^{\frac{k}{2}} du = \frac{1}{2} k(k+1)^\omega \lambda^{\frac{k}{2}} \leq C_0.$$

The second one can be estimated in the following way:

$$\begin{aligned} (k+1)^\omega \int_{\frac{k}{2}}^k \lambda^{k-u} (u+1)^{-\omega} du &\leq \int_{\frac{k}{2}}^k \lambda^{k-u} \left(\frac{k+1}{\frac{k}{2}+1} \right)^{-\omega} du \leq \\ &\leq C_1 \lambda^k \int_{\frac{k}{2}}^k \lambda^{-u} du = C_2 \lambda^k \left(\lambda^{-k} - \lambda^{-\frac{k}{2}} \right) = C_2 \left(1 - \lambda^{\frac{k}{2}} \right) \leq C_3. \end{aligned}$$

Here is the estimate for the second summand from (4.10):

$$(k+1)^\omega \sum_{u=k+1}^{\infty} \lambda^{u-k} (u+1)^{-\omega} \leq \sum_{u=k+1}^{\infty} \lambda^{u-k} < C_4.$$

□

Now we prove Theorem 1. We show how property $B_\omega(\mathbb{Z}^+)$ implies hyperbolicity. Let

$$K = 2r^2 M^2, \quad \lambda = (1 - (2rM)^{-1}), \quad P_l = X_l P X_{-l}, \quad Q_l = X_l (I - P) X_{-l}.$$

Using Theorem 3 it is easy to check that the first two conditions from the definition of hyperbolicity and inequalities from Remark 2.4 are satisfied. The uniform estimates of the norms of the projectors P_k and Q_k are due to Remark 2.3.

Now we show how hyperbolicity implies $B_\omega(\mathbb{Z}^+)$. Let $\|f\|_\omega < R$. We define a sequence y_k as follows:

$$y_k = \sum_{u=0}^k \Phi_{k,u} P_u f_u - \sum_{u=k+1}^{\infty} \Phi_{k,u} Q_u f_u.$$

Let K, λ be the numbers from the definition of hyperbolicity of the sequence \mathcal{A} . Then using the Lemma 4.8 we write this estimates:

$$\begin{aligned} (k+1)^\omega |y_k| &\leq R(k+1)^\omega \left(\sum_{u=0}^k (u+1)^{-\omega} \|\Phi_{k,u} P_u\| + \sum_{u=k+1}^{\infty} (u+1)^{-\omega} \|\Phi_{k,u} Q_u\| \right) \leq \\ &\leq RK(k+1)^\omega \left(\sum_{u=0}^k \lambda^{k-u} (u+1)^{-\omega} + \sum_{u=k+1}^{\infty} \lambda^{u-k} (u+1)^{-\omega} \right) < C. \end{aligned}$$

4.2. Pliss Theorem. Let $I = \mathbb{Z}$ and $\omega \geq 0$. We assume that the norms of A_k and A_k^{-1} are bounded.

Statement 4.9. *If a sequence \mathcal{A} have property $B_\omega(\mathbb{Z})$ then it is hyperbolic on \mathbb{Z}^+ and \mathbb{Z}^- with corresponding stable and unstable spaces S_k^+ , U_k^+ and S_k^- , U_k^- .*

Proof. Since we have property $B_\omega(\mathbb{Z})$ for the sequence \mathcal{A} , we also have properties $B_\omega(\mathbb{Z}^+)$ for its positive part $\{A_k\}_{k=0}^\infty$ and $B_\omega(\mathbb{Z}^-)$ for its negative part $\{A_k\}_{k=-\infty}^0$. Then the hyperbolicity on \mathbb{Z}^+ follows directly from the Maizel theorem. In particular this means that there exist stable and unstable subspaces S_k^+ and U_k^+ . Hyperbolicity on \mathbb{Z}^- also follows from Maizel theorem but it should be applied not to equations (2.1) and (2.2) but to the equations with inverted time:

$$\begin{aligned} x_k &= A_k^{-1} x_{k+1}, & k \in \mathbb{Z}^+, \\ x_{k+1} &= A_k^{-1} x_{k+1} - A_k^{-1} f_{k+1}, & k \in \mathbb{Z}^+. \end{aligned}$$

Thus hyperbolic sequence $\{A_{-k}^{-1}\}_{k=0}^{\infty}$ has spaces \tilde{S}_k and \tilde{U}_k . We denote $U_k^- = \tilde{S}_k$ and $S_k^- = \tilde{U}_k$ keeping in mind the sequence $\{A_k\}_{k=-\infty}^0$.

□

Remark 4.10. It is easy to see that Statement 4.1 is still correct for sequences on \mathbb{Z} under the assumptions of the previous statement. Spaces S_0^- and U_0^- play the role of spaces V_1 and V_2 .

Statement 4.11. *If a sequence \mathcal{A} have property $B_\omega(\mathbb{Z})$ then it is hyperbolic both on \mathbb{Z}^+ and \mathbb{Z}^- and spaces S_0^+ and U_0^- from Statement 4.9 are transverse.*

Proof. Let S_0^+ and U_0^- be not nontransverse. Then there exists a vector x such that $x \neq y_1 + y_2$, where $y_1 \in S_0^+$, $y_2 \in U_0^-$. We know that $S_0^+ \cap U_0^+ = 0$, $S_0^+ + U_0^+ = \mathbb{R}^n$ therefore x can be represented as $x = \zeta + \eta$ with $\zeta \in S_0^+$ and $\eta \in U_0^+$. Thus $\eta \neq z_1 + z_2$ for $z_1 \in S_0^+$, $z_2 \in U_0^-$. Take a sequence of numbers a_k whose entries equal 0 for negative indices and are in $(0, 1)$ for nonnegative. We construct a sequence θ_k that will drive us to a contradiction:

$$\theta_k = - \sum_{i=k+1}^{\infty} \Phi_{k,i} f_i, \quad k \in \mathbb{Z},$$

where

$$f_i = a_i(i+1)^{-\omega} \frac{\Phi_{i,0}\eta}{|\Phi_{i,0}\eta|}.$$

Vectors f_i belong to U_i^+ , $i \geq 0$, because $\Phi_{i,j}$ maps U_j^+ to U_i^+ by the hyperbolicity definition. The series from the definition of θ_k converges:

$$\begin{aligned} \left| \sum_{i=k+1}^{\infty} \Phi_{k,i} f_i \right| &\leq \sum_{i=\max(0,k)+1}^{\infty} |\Phi_{k,i} f_i| \leq \sum_{i=\max(0,k)+1}^{\infty} C|\eta| \lambda^{i-k} (i+1)^{-\omega} \leq \\ &\leq C|\eta| \sum_{l=1}^{\infty} \lambda^l \leq C_1 \end{aligned}$$

Recall that the sequence $\{\theta_k\}_{k \in \mathbb{Z}^+}$ belongs to $\mathcal{N}_\omega(\mathbb{Z}^+)$:

$$\begin{aligned} (k+1)^\omega \left| \sum_{i=k}^{\infty} \Phi_{k,i} f_i \right| &\leq (k+1)^\omega \sum_{i=\max(0,k)+1}^{\infty} |\Phi_{k,i} f_i| \leq \\ &\leq (k+1)^\omega \sum_{i=\max(0,k)+1}^{\infty} (i+1)^{-\omega} \left\| \Phi_{k,i}|_{U_i^+} \right\| |\eta| \leq C_2 |\eta| \sum_{i=\max(0,k)+1}^{\infty} \lambda^{i-k} \leq \\ &\leq C_2 |\eta| \sum_{l=1}^{\infty} \lambda^l \leq C_3. \end{aligned}$$

It is easy to see that the sequence θ_k is a solution of the inhomogeneous equation (2.2). Moreover, the following equality is satisfied

$$\theta_0 = - \sum_{i=1}^{\infty} \Phi_{0,i} f_i = C_4 \eta.$$

This means that for θ_0 the same thing as for η is true

$$(4.11) \quad \theta_0 \neq y_1 + y_2,$$

with $y_1 \in S_0^+$, $y_2 \in U_0^-$.

Because of Remark 4.10 there existss the only solution $\{\psi_k\} \in \mathcal{N}_\omega(\mathbb{Z})$ of inhomogeneous equation (2.2) with inhomogeneity f such that $\psi_0 \in U_0^-$. Every other solution of the inhomogeneous equation can be obtained adding a solution of the homogeneous equation. Thus $\{\psi_k\}_{k \in \mathbb{Z}} = \{X_k(\psi_0 - \theta_0) + \theta_k\}_{k \in \mathbb{Z}}$. So we have $\{X_k(\psi_0 - \theta_0)\}_{k \in \mathbb{Z}^+} = \{\psi_k\}_{k \in \mathbb{Z}^+} - \{\theta_k\}_{k \in \mathbb{Z}^+} \in \mathcal{N}_\omega(\mathbb{Z}^+)$ since $\mathcal{N}_\omega(\mathbb{Z}^+)$ is a linear space. From this we obtain that the vector $\psi_0 - \theta_0$ belongs to S_0^+ . Therefore if we denote $y_1 = \theta_0 - \psi_0$, $y_2 = \psi_0$ then we have $\theta_0 = y_1 + y_2$, what contradicts inequality (4.11). \square

Now we prove Theorem 2.

At first we show how the existence of property $B_\omega(\mathbb{Z})$ follows from the hyperbolicity on \mathbb{Z}^+ and \mathbb{Z}^- and transversality of $B^+(\mathcal{A})$ and $B^-(\mathcal{A})$. Fix a sequence $f \in \mathcal{N}_\omega(\mathbb{Z})$. Consider its positive and negative parts

$$f^+ = \{f_k\}_{k \in \mathbb{Z}^+}, \quad f^- = \{f_k\}_{k \in \mathbb{Z}^-}.$$

Since the sequence \mathcal{A} is hyperbolic on both \mathbb{Z}^+ and \mathbb{Z}^- , by the Maizel theorem its positive and negative parts $\mathcal{A}^+ = \{A_k\}_{k \in \mathbb{Z}^+}$ and $\mathcal{A}^- = \{A_k\}_{k \in \mathbb{Z}^-}$ have properties $B_\omega(\mathbb{Z}^+)$ and $B_\omega(\mathbb{Z}^-)$ correspondingly. Thus there exist solutions $\psi^+ \in \mathcal{N}_\omega(\mathbb{Z}^+)$ and $\psi^- \in \mathcal{N}_\omega(\mathbb{Z}^-)$ of equations (2.2) for $I = \mathbb{Z}^+$ and $I = \mathbb{Z}^-$ with inhomogeneities f^+ and f^- correspondingly. If $\psi_0^+ = \psi_0^-$ then the sequence ψ with

$$\psi_k = \psi_k^-, \quad k \leq 0, \quad \psi_k = \psi_k^+, \quad k > 0$$

is a solution of the inhomogeneous system (2.2) for $I = \mathbb{Z}$ and belongs to $\mathcal{N}_\omega(\mathbb{Z})$. If $\psi_0^+ \neq \psi_0^-$ then the solutions ψ^+ and ψ^- can be modified by solutions of the homogeneous systems: we show that there exist solutions $\phi^+ \in \mathcal{N}_\omega(\mathbb{Z}^+)$ and $\phi^- \in \mathcal{N}_\omega(\mathbb{Z}^-)$ of the homogeneous system (2.1) for $I = \mathbb{Z}^+$ and $I = \mathbb{Z}^-$ such that $\psi_0^+ + \phi_0^+ = \psi_0^- + \phi_0^-$. The last condition can be rewritten as

$$(4.12) \quad \psi_0^+ - \psi_0^- = \phi_0^- - \phi_0^+.$$

Recall that $B^+(\mathcal{A}) = S_0^+$ and $B^-(\mathcal{A}) = U_0^-$ since for a hyperbolic sequence solutions of the corresponding homogeneous linear system of difference equations are either tend to infinity with exponential speed or or tend to zero with exponential speed.

By assumption the spaces $B^+(\mathcal{A})$ and $B^-(\mathcal{A})$ are transverse so every vector from \mathbb{R}^d can be represented as a difference from the right hand side of (4.12). In particular we can obtain the left hand side of (4.12).

To obtain hyperbolicity and transversality from property B_γ we only need to use Statement 4.11 and the fact that $B^+(\mathcal{A}) = S_0^+$ and $B^-(\mathcal{A}) = U_0^-$.

5. APPLICATION OF THE GENERALIZATION OF DISCRETE ANALOG OF PLISS THEOREM

In this section we apply the generalized version of Pliss theorem for difference equations in shadowing theory.

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical

systems (see, for example, the monographs [13, 9]). In particular the connections between shadowing and structural stability are interesting.

It is well known that a structurally stable system has shadowing property and this property is Lipschitz (see [13]). Recently it was shown that Lipschitz shadowing implies structural stability (see [14]). Also structural stability follows from Hölder shadowing property under some additional assumptions (see [18]). Moreover it is known that structurally stable system has two-sided limit shadowing property but even the C^1 -interior of the set of diffeomorphisms having two-sided shadowing property (without prescribing the speed of convergence to zero of estimates on each step) not coincide with the set of structurally stable diffeomorphisms (see [12]).

We show that Lipschitz two-sided limit shadowing property is equivalent to structural stability.

5.1. Definitions. Let f be a homeomorphism of a metric space (M, dist) and consider a dynamical system that is generated by f .

Definition 3. We say that a sequence $\{x_k\}_{k \in \mathbb{Z}}$ of points of M is a d -pseudotrajectory of the dynamical system f if the following inequalities are satisfied

$$\text{dist}(x_{k+1}, f(x_k)) < d, \quad k \in \mathbb{Z}.$$

Let γ be a nonnegative real number.

Definition 4. We say that a sequence $\{x_k\}_{k \in \mathbb{Z}}$ of points of M is a γ -decreasing d -pseudotrajectory of the dynamical system f if the following inequalities are satisfied

$$\text{dist}(x_{k+1}, f(x_k)) < d(|k| + 1)^{-\gamma}, \quad k \in \mathbb{Z}.$$

Definition 5. We say that the homeomorphism f has Lipschitz two-sided limit shadowing property with exponent γ if there exist positive constants d_0, L such that for any γ -decreasing d -pseudotrajectory $\{x_k\}$ with $d \leq d_0$ there exists a point $p \in M$ such that

$$\text{dist}(x_k, f^k(p)) \leq Ld(|k| + 1)^{-\gamma}, \quad k \in \mathbb{Z}.$$

We write $f \in \text{LTSLmSP}(\gamma)$ in this case.

Remark 5.1. In [13], where some similar shadowing properties has been studied. It is shown that in the neighborhood of a hyperbolic set both L_p -shadowing and weighted shadowing (for a special choice of weights) are present.

5.2. Main results. A diffeomorphism f of a smooth manifold M is said to be structurally stable if there exists a neighborhood U of the diffeomorphism f in the C^1 -topology such that any diffeomorphism $g \in U$ is topologically conjugate to f .

Theorem 4. Let $\gamma \geq 0$ and f be a diffeomorphism of a closed Riemannian manifold M . Then f is structurally stable iff $f \in \text{LTSLmSP}(\gamma)$.

5.3. Lipschitz two-sided limit shadowing property implies structural stability. We use one well-known result of R. Mane. Let M be a closed Riemannian manifold and f be a diffeomorphism of M . We denote the tangent space to the manifold M at a point p by $T_p M$. Fix a point $p \in M$ and consider two linear subspaces of $T_p M$:

$$\begin{aligned} B^+(p) &= \{v \in T_p M \mid |Df^k(p)v| \rightarrow 0, \quad k \rightarrow +\infty\}, \\ B^-(p) &= \{v \in T_p M \mid |Df^k(p)v| \rightarrow 0, \quad k \rightarrow -\infty\}. \end{aligned}$$

Definition 6. We say that for a diffeomorphism f the analytical transversality condition is satisfied at a point p if

$$B^+(p) + B^-(p) = T_p M.$$

Theorem (Mañé, [8]). *Diffeomorphism f is structurally stable iff the analytical transversality condition is satisfied at every point p of M .*

At first we prove one simple lemma

Lemma 5.2. *If for a sequence $\{w_k\}_{k \in \mathbb{Z}}$ from \mathcal{N}_γ there exists a constant Q such that for any integer $N > 0$ there exists a sequence $\{v_k^N\}_{k \in [-N, N]}$ of vectors from \mathbb{R}^d satisfying equalities*

$$(5.1) \quad v_{k+1}^N = A_k v_k^N + w_{k+1}, \quad k \in [-N, N-1],$$

and inequalities $|v_k^N| (|k| + 1)^\gamma \leq Q$, $k \in [-N, N]$ then there exists a sequence $\{v_k\}_{k \in \mathbb{Z}}$ such that it satisfy the same inequalities (5.1) for every integer k and $\|\{v_k\}_{k \in \mathbb{Z}}\|_\gamma \leq Q$.

Proof. To obtain a sequence needed we use a diagonal procedure (we take $v_k^N = 0$, $k \notin [-N, N]$) and pass to a limit in inequalities (5.1). Despite the convergence is not uniform in general the sequence we get as a result has all the necessary properties. \square

Statement 5.3. *Let f be a diffeomorphism of a closed Riemannian manifold and let $\gamma > 0$. If $f \in LTS\text{Lm}SP(\gamma)$ then f is structurally stable.*

Proof. Using Theorem 2 we show that if we have Lipschitz two-sided limit shadowing property then the analytical transversality condition is satisfied at every point. After that we just apply the Mane theorem.

Fix a point $p \in M$, denote $p_k = f^k(p)$ and define linear isomorphisms $A_k = Df(p_k)$ for $k \in \mathbb{Z}$. We denote a ball in M with a radius r and a center x by $B(r, x)$ and a ball in $T_x M$ with radius r and center 0 by $B_T(r, x)$.

The fact that the norms of all A_k and A_k^{-1} are bounded follows from the compactness of the manifold. We prove that under our assumptions property $B_\gamma(\mathbb{Z})$ is satisfied for the sequence of matrices A_k . After that we will be able to use Theorem 2.

Let $\exp_x : T_x M \rightarrow M$ be a standard exponential mapping. There exists a $r > 0$ such that for any point $x \in M$ the mapping \exp_x is a diffeomorphism of a ball $B_T(r, x)$ onto its image and \exp_x^{-1} is a diffeomorphism of a ball $B(r, x)$ onto its image. Moreover, we may assume that the smallness of r allows us to write the following estimates for relations between distances in the manifold and in a tangent space:

if $v, w \in B_T(r, x)$ then

$$(5.2) \quad \text{dist}(\exp_x(v), \exp_x(w)) \leq 2|v - w|;$$

if $y, z \in B(r, x)$ then

$$(5.3) \quad |\exp_x^{-1}(y) - \exp_x^{-1}(z)| \leq 2\text{dist}(y, z).$$

Consider mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : T_{p_k} M \rightarrow T_{p_{k+1}} M.$$

From the well-known properties of an exponential mapping we deduce that $D \exp_x(0) = \text{Id}$; therefore

$$DF_k(0) = Df(p_k).$$

Since M is compact for any $\varepsilon > 0$, we can find a $\delta > 0$ such that if $|v| \leq \delta$, then for $g_k(v) = F_k(v) - A_k v$ the following inequality is satisfied

$$(5.4) \quad |g_k(v)| \leq \varepsilon |v|.$$

Let L, d_0 be the constants from the definition of $LTSLmSP(\gamma)$.

We prove that for any sequence of vectors $\{z_k\}_{k \in \mathbb{Z}} \in \mathcal{N}_\gamma$ satisfying $\|\{z_k\}_{k \in \mathbb{Z}}\|_\gamma < 1$ there exists a sequence $\{v_k\}_{k \in \mathbb{Z}} \in \mathcal{N}_\gamma$ that is a solution of equations

$$(5.5) \quad v_{k+1} = A_k v_k + z_{k+1}, \quad k \in \mathbb{Z}.$$

After this if we use Theorem 2 then we obtain that the analytical transversality condition is satisfied at the point p .

We show that the conditions of Lemma 5.2 are satisfied.

We fix natural N , small positive d , and define vectors a_k :

$$(5.6) \quad a_{-N} = 0, \quad a_{k+1} = A_k a_k + z_{k+1}, \quad k \in [-N, N-1].$$

Now we assume that d is small enough so that all the points of M that appear belong to the corresponding balls $B(r, p_k)$ and all tangent vectors from $T_{p_k} M$ that appear belong to the corresponding balls $B_T(r, p_k)$.

We define a sequence $\xi_k \in M$ in the following way: let $\xi_k = \exp_{p_k}(da_k)$ for $|k| \leq N$, $\xi_{N+k} = f^k(\xi_N)$ for $k > 0$ and $\xi_{-N+k} = f^k(\xi_{-N})$ for $k < 0$.

We estimate $\text{dist}(f(\xi_k), \xi_{k+1})$ for $k \in [-N, N-1]$.

Since

$$\exp_{p_{k+1}}^{-1}(f(\xi_k)) = F_k(da_k) = A_k(da_k) + g_k(da_k)$$

and

$$\exp_{p_{k+1}}^{-1}(\xi_{k+1}) = da_{k+1} = d(A_k a_k + z_{k+1}),$$

after use of estimates (5.3) and (5.4) we obtain

$$\begin{aligned} \text{dist}(f(\xi_k), \xi_{k+1}) &\leq 2|F_k(da_k) - da_{k+1}| = \\ &= 2|g_k(da_k) - dz_{k+1}| \leq 4d(|k| + 1)^{-\gamma}. \end{aligned}$$

For $k \notin [-N, N-1]$ the distance $\text{dist}(f(\xi_k), \xi_{k+1})$ equals 0. Thus the sequence ξ_k is a γ -decreasing $4d$ -pseudotrajectory. Without loss of generality we can assume that $4d < d_0$. Since $f \in LTSLmSP(\gamma)$, there exists a sequence $y_k \in M$ such that

$$\text{dist}(\xi_k, y_k) \leq 4Ld(|k| + 1)^{-\gamma}.$$

Let $t_k = \exp_{p_k}^{-1}(y_k)$. Then

$$t_{k+1} = F_k(t_k) = A_k t_k + g_k(t_k).$$

Using the definition of a_k it is easy to see that

$$\begin{aligned} |t_k| &\leq 2 \text{dist}(y_k, p_k) \leq 2 \text{dist}(y_k, \xi_k) + 2 \text{dist}(\xi_k, p_k) \leq \\ &\leq 8Ld(|k| + 1)^{-\gamma} + 4d|a_k| \leq \text{const} \left(N, L, \sup_{p \in M} \|Df(p)\| \right) d, \quad k \in [-N, N]. \end{aligned}$$

This means that $|g_k(t_k)|$ can be made as small as we need only by decreasing of d . Denote

$$\begin{aligned} b_k &= \Phi_{k, -N} t_{-N}, \quad k \in [-N, N], \\ c_k &= t_k - b_k, \quad k \in [-N, N]. \end{aligned}$$

Then

$$\begin{aligned} c_{-N} &= 0, \quad c_{k+1} = A_k c_k + g_k(t_k), \quad k \in [-N, N-1], \\ |c_k| &= |(\Phi_{k,-N} t_{-N} + g_k(t_k) + A_k g_{k-1}(t_{k-1}) + \dots) - \Phi_{k,-N} t_{-N}| < \\ &< d(|k| + 1)^{-\gamma}, \quad k \in [-N, N-1]. \end{aligned}$$

Now we show that this is the sequence we have looked for:

$$\begin{aligned} w_k &= a_k - \frac{b_k}{d}, \quad k \in [-N, N-1], \\ w_k &= 0, \quad k \notin [-N, N-1]. \end{aligned}$$

The fact that this sequence is a solution of equations (5.5) is obvious. To estimate its norm in the space \mathcal{N}_γ by a number independent of N we write the following:

$$\begin{aligned} \left| a_k - \frac{b_k}{d} \right| &= \frac{1}{d} |da_k - b_k| = \frac{1}{d} |(\exp_{p_k}^{-1}(\xi_k) - \exp_{p_k}^{-1}(y_k)) + c_k| \leq \\ &\leq (8Ld(|k| + 1)^{-\gamma} + d(|k| + 1)^{-\gamma}) = (8L + 1)(|k| + 1)^{-\gamma}. \end{aligned}$$

□

5.4. Structural stability implies Lipschitz two-sided limit shadowing property. We use the method from [13] to prove that structural stability implies Lipschitz two-sided limit shadowing property. Let H_k , $k \in \mathbb{Z}$ be a sequence of subspaces of \mathbb{R}^d . Consider a sequence of linear mappings $\mathcal{A} = \{A_k : H_k \rightarrow H_{k+1}\}$.

Definition 7. We say that a sequence \mathcal{A} has property (C) with constants $N > 1$ and $\lambda \in (0, 1)$ if for any integer k there exist projections P_k, Q_k such that if $S_k = P_k H_k$ and $U_k = Q_k H_k$ then the following conditions are satisfied:

- $P_k + Q_k = Id$, $\|P_k\|, \|Q_k\| \leq N$;
- $A_k S_k \subset S_{k+1}$ and $\|A_k|_{S_k}\| \leq \lambda$;
- If $U_{k+1} \neq \{0\}$ then there exists a linear mapping $B_k : U_{k+1} \rightarrow H_k$ such that

$$B_k U_{k+1} \subset U_k, \quad \|B_k\| \leq \lambda, \quad A_k B_k|_{U_{k+1}} = I.$$

Theorem 5. Let $\gamma > 0$ and let \mathcal{A} have property (C) with constants $N > 1$ and $\lambda \in (0, 1)$. Consider a sequence of mappings $f_k : H_k \rightarrow H_{k+1}$ of form $f_k(v) = A_k v + w_{k+1}(v)$. Suppose that there exist constants $\kappa, \Delta > 0$ such that the following inequalities are satisfied:

$$|\omega_k(v) - \omega_k(v')| \leq \kappa |v - v'|, \quad |v|, |v'| \leq \Delta, \quad k \in \mathbb{Z};$$

$$\kappa N_1 < 1,$$

with

$$N_1 = N \frac{1 + \lambda}{1 - \lambda}.$$

Denote

$$L = \frac{N_1}{1 - \kappa N_1}.$$

Then if

$$\|\{f_k(0)\}\|_\gamma \leq d \leq d_0, \quad k \in \mathbb{Z},$$

with

$$d_0 = \frac{\Delta}{L},$$

then there exists a sequence $v_k \in H_k$ such that $f_k(v_k) = v_{k+1}$ and $\|\{v_k\}\|_\gamma \leq Ld$.

Proof. Consider an operator $G : \mathcal{N}_\gamma(\mathbb{Z}) \rightarrow (\mathbb{R}^d)^\mathbb{Z}$ of the form

$$G(z) = \hat{g}_1(z) + \hat{g}_2(z),$$

with

$$\begin{aligned} (\hat{g}_1(z))_k &= \sum_{u=-\infty}^k A_{k-1} \dots A_u P_u z_u, \\ (\hat{g}_2(z))_k &= - \sum_{u=k+1}^{\infty} B_k \dots B_{u-1} P_u z_u. \end{aligned}$$

We prove that the operator G maps $\mathcal{N}_\gamma(\mathbb{Z})$ to $\mathcal{N}_\gamma(\mathbb{Z})$ and is bounded. We prove that $(|k| + 1)^\gamma (G(x))_k \leq C$ for $k \geq 0$ (for $k \leq 0$ the proof is analogous). We represent $G(z)$ in the following form

$$G(z) = g_1(z) + g_2(z) + g_3(z)$$

with

$$\begin{aligned} (g_1(z))_k &= \sum_{u=-\infty}^0 A_{k-1} \dots A_u P_u z_u, \\ (g_2(z))_k &= \sum_{u=0}^k A_{k-1} \dots A_u P_u z_u, \\ (g_3(z))_k &= - \sum_{u=k+1}^{\infty} B_k \dots B_{u-1} P_u z_u. \end{aligned}$$

Lemma 4.8 allows us to estimate $(k+1)^\gamma |(g_2(z))_k + (g_3(z))_k|$. It remains to estimate only $(k+1)^\gamma |(g_1(z))_k|$:

$$\begin{aligned} (k+1)^\gamma (g_1(z))_k &\leq (k+1)^\gamma \sum_{u=-\infty}^0 \lambda^{k-u} (|u| + 1)^{-\gamma} \leq \\ &\leq (k+1)^\gamma \lambda^k \sum_{u=0}^{\infty} \lambda^u \leq C_1. \end{aligned}$$

The rest of the proof is fully analogous to the proof of Theorem 1.3.1 from [13]. \square

The next statement is proved in [13] (Lemma 2.2.16).

Statement 5.4. *Let f be a structurally stable diffeomorphism of the closed Riemannian manifold M . Then for each point p of M there exist spaces $S(p), U(p) \subset T_p M$ such that the following holds:*

- (1) $S(p) \oplus U(p) = T_p M$;
- (2) $Df(p)S(p) \subset S(f(p))$, $Df^{-1}(p)U(p) \subset U(f^{-1}(p))$;
- (3) There exist constants $C_1 > 0$ and $\lambda_1 \in (0, 1)$ such that

$$|Df^k(p)v| \leq C_1 \lambda_1^k |v|, \quad v \in S(p), \quad k \geq 0,$$

$$|Df^{-k}(p)v| \leq C_1 \lambda_1^k |v|, \quad v \in U(p), \quad k \geq 0;$$

- (4) There exists a constant N such that if $P(p)$ and $Q(p)$ are complementary projectors corresponding to $S(p)$ and $U(p)$ then

$$\|P(p)\|, \|Q(p)\| \leq N;$$

- (5) For any $\beta > 0$ and a natural T there exists a number $\alpha > 0$ such that if $z, p \in M$, $q = f^T(p)$, $y = f^{-T}(z)$ and $\text{dist}(z, q) < \alpha$ then there exist linear isomorphism $\Pi(p, x) : T_z M \rightarrow T_z M$ such that

$$\|\Pi - I\| \leq \beta, \quad \Pi(p, z)(D(\exp_z^{-1}(q))Df^{-T}(p)S(p)) \subset S(z)$$

and linear isomorphism $\Theta(p, z) : T_p M \rightarrow T_p M$ such that

$$\|\Theta - I\| \leq \beta, \quad \Theta(p, z)(D(\exp_p^{-1}(y))Df^{-T}(z)U(z)) \subset U(p).$$

Lemma 5.5. *Let f be a diffeomorphism of M and T be a natural number. If $f^T \in LTS\text{LmSP}(\gamma)$ then also $f \in LTS\text{LmSP}(\gamma)$.*

Proof. Fully analogous to the proof of lemma 1.1.3 from [13]. \square

Theorem 6. *Let f be a structurally stable diffeomorphism of the closed Riemannian manifold M . Then $f \in LTS\text{LmSP}(\gamma)$ for any $\gamma \geq 0$.*

Proof. This theorem is proved in the same way as Theorem 2.2.7 from [13]. We choose T such that

$$\mu = C\lambda_1^T < 1.$$

Lemma 5.5 shows that to prove that $f \in LTS\text{LmSP}(\gamma)$ it is enough to show that $f' = f^T \in LTS\text{LmSP}(\gamma)$. We use Statement 5.4 for $C = 1$, $\lambda_1 = \mu$ and $T = 1$.

We take $\nu_0 \in (0, 1)$ such that $\lambda = (1 + \nu_0)^2 \mu < 1$. For this λ and the number N from part 4 of the statement 5.4 we take the corresponding N_1 from the conditions of Theorem 5 and find $\kappa > 0$ such that $\kappa N_1 < 1$. We denote

$$K = \max(N, \max_{p \in M} \|Df(p)\|, \max_{p \in M} \|Df^{-1}(p)\|).$$

We find a number $\nu \in (0, \nu_0)$ such that

$$2K(2K + 1)\nu < \kappa/2.$$

Let c be a radius such that for each $p \in M$ the mapping \exp_p is a diffeomorphism of the ball $E_{2c}(p) \subset T_p M$ onto its image. We take a number $d' < c$ such that for all point $x, y \in M$ for $\text{dist}(x, y) \leq d'$ and $y' = \exp_x^{-1}(y)$ the following inequalities are satisfied

$$|D \exp_x^{-1}(y)v| \leq (1 + \nu), \quad |D \exp_x(y')v| \leq (1 + \nu).$$

For $\beta = \nu$ we choose a number d'' such that part 5 of Statement 5.4 is satisfied. We assume that $d' \leq d''$.

Let x_k be a γ -decreasing d -pseudotrajectory for $d \leq d'$.

Consider the mappings $f_k : T_{x_k} \rightarrow T_{x_{k+1}}$ defined as follows

$$f_k(v) = \exp_{x_{k+1}}^{-1} \circ f \circ \exp_{x_k}(v).$$

Then the mappings f_k are defined on $E_{d'}(x_k)$. It is easy to see that

$$|f_k(0)| \leq d(|k| + 1)^{-\gamma}.$$

Now we introduce the following notations:

$$p = x_k, \quad \hat{p} = f(x_k) = f(p), \quad z = x_{k+1},$$

$$\tilde{z} = f^{-1}(x_{k+1}) = f^{-1}(z), \quad \tilde{z}' = \exp_p^{-1}(\tilde{z}) = \exp_{x_k}^{-1}(f^{-1}(x_{k+1})).$$

Denote $J_k = Df_k(0)$. Then

$$J_k = D \exp_z^{-1}(\hat{p}) Df(p) = D \exp_{x_{k+1}}^{-1}(f(x_k)) Df(x_k).$$

Denote

$$F_k = Df(\tilde{z})D \exp_p(\tilde{z}') = Df(f^{-1}(x_{k+1}))D \exp_p(\exp_{x_k}^{-1}(f^{-1}(x_{k+1})))$$

and

$$A_k^s = \Pi(p, z)J_k P(p), \quad A_k^u = F_k \Theta^{-1}(p, z)Q(p), \quad A_k = A_k^s + A_k^u.$$

It is proved in [13] that f_k can be represented in the form

$$f_k(v) = A_k v + (J_k - A_k)v + \chi(v),$$

with $|\omega_k(v) - \omega_k(v')| \leq \kappa |v - v'|$ and that the sequence $\{A_k\}$ has property (C).

We take $\Delta = d'$ and use Theorem 5. Then there exist numbers d_0, L such that for $\|f_k(0)\|_\gamma \leq d \leq d_0$ there exists a sequence $v_k \in T_{x_k}$ such that

$$\|\{v_k\}\|_\gamma \leq Ld, \quad f_k(v_k) = v_{k+1}.$$

Denote $x = \exp_{x_0}(v_0)$. Then

$$f^k(x) = \exp_{x_k}(v_k).$$

Thus

$$\text{dist}(x_k, f^k(x)) = |v_k| \leq Ld(|k| + 1)^{-\gamma}.$$

□

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